Become Friends With Logarithms

Logarithms are actually very easy once you get used to how they behave. After that they are as predictable as yesterday's weather. All it takes is practice.

What is a logarithm? A logarithm is an exponent. Specifically, logarithms are written "$\log_b x = y$" and read as "the log, base b, of x is y". It *really* means "base b, taken to the power y, gives x". Hence,

$$\log_b x = y \Leftrightarrow b^y = x$$

So, for instance, if you had an unknown variable as an exponent, you'd work out a logarithm to find that exponent's value. Here's an easy example: $2^x = 8$. In this case we know from past experience that $x = 3$, since $2^3 = 8$. Hence, "3 is the logarithm, base 2, of 8", or $3 = \log_2 8$.

In this set, you take a known expression that should be obviously true, and convert it into its equivalent logarithm form. This will give you practice to see where the proper places are for each number in the logarithm form.

Example: Given $4^2 = 16$, you'd rewrite this as $\log_4 16 = 2$.

$2^4 = 16$	$5^3 = 125$
$3^2 = 9$	$7^0 = 1$
$10^3 = 1000$	$8^{1/3} = 2$
$25^{1/2} = 5$	$2^{-1} = \frac{1}{2}$
$10^{-2} = 0.01$	$3^{-3} = \frac{1}{27}$

Now we go backwards, so to speak. Convert these true logarithms back into their equivalent exponential format. At first you may not see that each given logarithm is true, but if you rewrite it as the more familiar exponential form, hopefully this will help you see how logs are properly written.

Example: $\log_6 36 = 2$ is a true logarithm statement. Trust me. If you rewrite this as $6^2 = 36$, perhaps you will agree now, too.

$\log_2 32 = 5$	$\log_9 81 = 2$	$\log_2 \frac{1}{4} = -2$
$\log_5 25 = 2$	$\log_{16} 4 = \frac{1}{2}$	$\log_{49} 7 = \frac{1}{2}$
$\log_{10} 10 = 1$	$\log_6 1 = 0$	$\log_8 2 = \frac{1}{3}$

- ***Practice Set III – Find the error!***

In this set, you have logarithms that seem to have all the right numbers but they are in the wrong places. You must re-arrange them so that the logarithm is corrected.

Example: $\log_2 3 = 8$ is incorrect. Literally translated into exponentials, this would read as $2^8 = 3$, which is obviously false (2 to the 8th power is 256, which is not equal to 3, of course). But, you know that $2^3 = 8$, so the corrected logarithm should be $\log_2 8 = 3$

$\log_5 2 = 25$	$\log_{25} \frac{1}{2} = 5$	$\log_7 1 = 7$
$\log_4 0 = 1$	$\log_{10} 2 = 100$	$\log_4 (-1) = \frac{1}{4}$
$\log_{64} 8 = 2$	$\log_3 5 = 125$	$\log_3 3 = 27$

- ***Practice Set IV – Find the Unknown.***

In each logarithm is an unknown value x. Determine its value. You may have to rewrite back into exponential form, but with practice you should be able to determine x without having to do this middle step.

Example: $\log_7 49 = x$. Rewrite as $7^x = 49$, which implies $x = 2$.

$\log_x 25 = 2$	$\log_4 2 = x$	$\log_{12} x = 1$
$\log_6 x = 2$	$\log_3 x = -1$	$\log_{10} 0.001 =$
$\log_8 64 = x$	$\log_9 1 = x$	$\log_x \frac{1}{125} = -3$

Note: In many books you will often see expressions like $\log_4 16$. In this case simply set it equal to a variable x and perform the steps as in the above set. You now have $\log_4 16 = x$.

Note: Square roots are equivalent to the $\frac{1}{2}$ power, cube roots are equivalent to the $\frac{1}{3}$ power, and so forth. Also recall that negative powers act as reciprocals, so that $3^{-1} = \frac{1}{3}, 3^{-2} = \frac{1}{9}$, et cetera. Many people stumble on these because they may have forgotten the basic rules of exponents. For example, $\log_5 \sqrt{5} = \frac{1}{2}$ and $\log_6 \frac{1}{36} = -2$. These are both true. Do you see why? If not, re-arrange into exponential form and see if that helps.

Logarithms and exponentials "undo" one another, just as squares and square roots undo one another. In algebra, if we need to "undo" an exponential, we apply a logarithm. Equivalently, if we need to "undo" a logarithm, we apply an exponential (usually we just re-write as an exponential). This leads to a handful of useful properties: (Remember, base b is always positive and $\neq 1$)

$\log_b b^x = x$. In this case, the "$\log_b$" cancels the base b, and leaves x all by itself.
$b^{\log_b x} = x$. This is a restatement of the above property. "Bases cancel logs".

Properties, continued.

$\log_b b = 1$, for all bases b. Rewrite into exponential form and this will be obvious. It is a corollary of the first property.
$\log_b 1 = 0$. This is true because $b^0 = 1$ for all bases b.

Algebraic properties of logarithms. There are three, and you must know these. This is how you actually manipulate logarithms as an algebraic operation.

I. $\log_b(xy) = \log_b x + \log_b y$

II. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

III. $\log_b x^n = n\log_b x$

All three properties have to do with how exponents behave when you multiply or divide terms with a common base.

Lastly, we note a couple of conventions:

"log" without a base written is understood to be $\log_{10}$.
"ln" is shorthand for the natural logarithm, which is the $\log_e$.

- ***Practice Set V – Expansions of Logarithms.***

Example: Expand $\log_2\left(\dfrac{4x^2y}{\sqrt{z}}\right)$. By rule II, we have $\log_2 4x^2y - \log_2 z^{1/2}$ (Do you see where the ½ power of z came from?). By Rule I we can break up the first term and get

$$\log_2 4x^2y - \log_2 z^{1/2} = \log_2 4 + \log_2 x^2 + \log_2 y - \log_2 z^{1/2}.$$

Now we can apply Rule III, and simplify the first term (remember, $\log_2 4 = 2$) to get

$$2 + 2\log_2 x + \log_2 y - \tfrac{1}{2}\log_2 z$$

Try these:

$\log_3(9xy^2)$

$\log_4(xy^2\sqrt{z})$

$\log_5\left(\dfrac{x}{25y}\right)$

$\log\left(\dfrac{100x^2}{y^3}\right)$

$\log_6(6x^2y)^3$

$\log\left(\dfrac{10(x-1)^2}{yz^2}\right)$

- ***Practice Set VI – Contraction of Logarithms***

Now go the other way – combine these individual logs into one big logarithm!

Example: $3+2\log_2 x+\frac{1}{2}\log_2 y-\log_2 z$. By rule II we can bring up the coefficients and turn them into powers and get $3+\log_2 x^2+\log_2\sqrt{y}-\log_2 z$. Recall that $3=\log_2 8$ and we have $\log_2 8+\log_2 x^2+\log_2\sqrt{y}-\log_2 z$. By rules I and II we get $\log_2\left(\frac{8x^2\sqrt{y}}{z}\right)$.

$2\log_3 x+3\log_3 y-\log_3 z$

$1+\log_4 x-\frac{1}{2}\log_4 y$

$2(\log x+3\log y)-5\log z$

$\frac{1}{2}(\log_2 x-(2\log_2 y+3\log_2 z))$

$2+\log x-\log y-2\log z$

$\frac{1}{3}+\log_3 x-2\log_3 y+3\log_3 z$

- ***Practice Set VII – Algebra With Logarithms***

In the equation $2^x=8$, it's pretty obvious that x = 3. You can essentially figure the answer by some educated guesses. What about the case of $2^x=6$? In this case we use logarithms to "undo" the exponential – i.e. get the x down from its perch. We take logarithms of both sides, use rule III and use the change-of-base rule to get our result.

Example: Solve $2^x=6$

Take logs of both sides (just log will do, it can be any base): $\log 2^x=\log 6$
Use rule III to bring down the exponent: $x\log 2=\log 6$.
Divide by log 2 and we get $x=\frac{\log 6}{\log 2}\approx 2.585...$
Check: $2^{2.585}=6.00015...$, which is close enough.

Use other rules of normal algebra as you see fit.

$3^x=7$

$5^x=12$

$4^{2x}=9$

$2(3^x)=12$

$1+2^x=8$

$7^{x+2}=20$

$5^{2x+3}=15$

$1+4(2^{x+1})=10$

$2^{x+1}=3^x$

$5^{x+1}+2=12$

$8^x=16$

$10^{4x-3}=1{,}000$